COMPLETE SYSTEMS OF PARTICULAR SOLUTIONS IN SHALLOW-SHELL THEORY

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In solving boundary value problems generally, and boundary value problems of shallow-shell theory in particular, it is necessary to have available a set of special solutions of the equations of this theory – either it is a fundamental solution or a complete system of particular solutions adapted for the considered domain. Precisely because of this circumstance, the overwhelming majority of distinct particular results obtained in recent years either refer to spherical or cylindrical shells, i. e. to those shells for which complete systems of particular solutions are known (see [1, 2], say).

The author of [3] obtained general representations of the solutions of the equations of shallow shell theory when the coefficients in these equations are analytic functions of the coordinates. In the Vekua representations the kernels are defined of solutions of some two-dimensional Volterra-type integral equations. If the coefficients in the equations of shallow-shell theory are assumed constant [4], the analysis is simplified somewhat.

For this case we can indicate a transformation scheme which would naturally reduce the solution to the Vekua type representations (1.2), where their kernels would be defined explicitly.

On the basis of these considerations, various complete systems of solutions as well as the fundamental solution of the shallow-shell theory equations are constructed below.

1. The equations of shallow-shell theory can be represented in the following equivalent form:

$$\frac{\partial^4 F}{\partial z^2 \partial \zeta^2} - \frac{\partial^2 F}{\partial z^2} - 2\delta \frac{\partial^2 F}{\partial z \partial \zeta} - \frac{\partial^2 F}{\partial \zeta^2} = f(z, \zeta) \qquad (1.1)$$

$$F = F_1 + iF_2, \quad F_1(z, \zeta) = U(x, y), \quad F_2(z, \zeta) = \frac{1}{\varepsilon^*} w(x, y)$$

$$z = \xi + i\eta = \frac{\beta \sqrt{i}}{a}(x + iy), \quad \zeta = \xi - i\eta = \frac{\beta \sqrt{i}}{a}(x - iy), \quad \beta = \frac{\sqrt{\varepsilon(1-\alpha)}}{4}$$

$$\varepsilon = \frac{a^2}{Rh}\sqrt{12(1-\mu^2)}, \quad \varepsilon^* = \frac{\sqrt{12(1-\mu^2)}}{Eh^2}, \quad \alpha = \frac{R}{R_1}, \quad |\alpha| \leqslant 1, \quad \delta = \frac{1+\alpha}{1-\alpha}$$

Here R, R_1, h and a are the principal radii of curvature, thickness, and characteristic linear dimension of the shell; E, μ are the Young's modulus and Poisson's ratio of the shell material, respectively; U, w are the stress function and deflection in the middle surface, respectively; $f(z, \zeta)$ is the right side in (1.1) which can be understood to be a quantity proportional to the loading or a temperature term; x and y are Cartesian coordinates on the shell surface.

It can be shown that the general representation of all regular solutions of (1.1) for $f(z, \zeta) = 0$ is

$$F(z, \zeta) = \varphi_0(z) \operatorname{ch}(\zeta - \zeta_0) + \psi_0(\zeta) \operatorname{ch}(z - z_0) - \qquad (1.2)$$

$$- \sum_{k=0}^{1} \int_{z_0}^{z} \varphi_k(t) \frac{\partial}{\partial t} G_k(z - t, \zeta - \zeta_0)\, dt - \sum_{k=0}^{1} \int_{\zeta_0}^{\zeta} \psi_k(\tau) \frac{\partial}{\partial \tau} G_k(z - z_0, \zeta - \tau)\, d\tau$$

Here $\varphi_k(z)$ and $\psi_k(\zeta)$ are arbitrary analytic functions of their arguments

$$G_1(z-t,\zeta-\tau)=G_1(\zeta-\tau,z-t)=\sum_{k=0}^{\infty}\frac{(z-t)^{k+1}}{(k+1)!}g_k(\zeta-\tau) \qquad (1.3)$$

$$G_0(z-t,\zeta-\tau)=G_0(\zeta-\tau,z-t)=\frac{\partial^2}{\partial z\partial\zeta}G_1(z-t,\zeta-\tau)$$

The functions $g_k(\zeta)$ can be represented in different forms.

a. As recurrent relationships

$$g_{2k+2}(\zeta)=\operatorname{sh}\zeta+\int_0^{\zeta}\operatorname{sh}(\zeta-\tau)\{2\delta g'_{2k+1}(\tau)+g''_{2k}(\tau)\}\,d\tau \qquad (k=0,1,\ldots)$$

$$g_{2k+3}(\zeta)=\int_0^{\zeta}\operatorname{sh}(\zeta-\tau)\{2\delta g'_{2k+2}(\tau)+g''_{2k+1}(\tau)\}\,d\tau \qquad (1.4)$$

$$g_0(\zeta)=\operatorname{sh}\zeta, \qquad g_1(\zeta)=\delta\zeta\operatorname{sh}\zeta$$

b. As series

$$g_{2k}(\zeta)=\sum_{s=0}^{\infty}\frac{\zeta^{2s+1}}{(2s+1)!}a_{k,s}, \quad a_{k,s}=(k+s)!\sum_{j=0}^{\min(k,s)}\frac{(2\delta)^{2j}}{(k-j)!\,(s-j)!\,(2j)!} \qquad (1.5)$$

$$g_{2k+1}(\zeta)=\sum_{s=0}^{\infty}\frac{\zeta^{2s+2}}{(2s+2)!}b_{k,s}, \quad b_{k,s}=(k+s+1)!\sum_{s=0}^{\min(k,s)}\frac{(2\delta)^{2j+1}}{(k-j)!\,(s-j)!\,(2j+1)!}$$

c. As the product of exponentials and polynomials

$$g_k(\zeta)=e^{\zeta}P_k(\zeta)+e^{-\zeta}Q_k(\zeta) \qquad (1.6)$$

Here $P_k(\zeta)$ and $Q_k(\zeta)$ are known polynomials of ζ of degree k.

The series (1.3) converge absolutely for any finite z and ζ.

If the right side $f(z,\zeta)$ in (1.1) is nonzero, then a member [3]

$$F^*(z,\zeta)=\int_{z_0}^{z}dt\int_{\zeta_0}^{\zeta}G_1(z-t,\zeta-\tau)f(t,\tau)\,d\tau \qquad (1.7)$$

appears in (1.2).

By virtue of Eq. (1.1) the kernels G_0 and G_1 are the solutions of this equation in arguments z, ζ and t, τ. G_1 is the Riemann function of the given equation.

The kernels $G_0(z,\zeta)$ and $G_1(z,\zeta)$ can be expressed as contour integrals. These formulas are of the form

$$G_k(z,\zeta)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}e^{p\zeta}G_k^*(p,z)\,dp, \quad \operatorname{Re}c>1 \qquad (k=0,1)$$

$$G_1^*(p,z)=\frac{1}{2pq}\left\{\exp\left[\frac{pz(\delta+q)}{p^2-1}\right]-\exp\left[\frac{pz(\delta-q)}{p^2-1}\right]\right\} \qquad (1.8)$$

$$G_0^*(p,z)=p\frac{\partial}{\partial z}G_1^*(p,z), \qquad q=\sqrt{p^2+\delta^2-1}$$

2. Let us introduce the functions

$$\Phi(z,\zeta)=\mathring{L}_{z,\zeta}\{\varphi(z)\}=\varphi(z)\operatorname{ch}(\zeta-\zeta_0)-\int_{z_0}^{z}\frac{\partial}{\partial t}G_0(z-t,\zeta-\zeta_0)\varphi(t)\,dt$$

$$\Phi^*(z,\zeta) = \overset{\circ}{L}_{\zeta,z}\{\varphi^*(\zeta)\} = \varphi^*(\xi)\,\mathrm{ch}\,(z-z_0) - \int_{\zeta_0}^{\zeta} \frac{\partial}{\partial\tau} G_0(z-z_0,\zeta-\tau)\,\varphi^*(\tau)\,d\tau$$

$$\Psi(z,\zeta) = L^1_{z,\zeta}\{\psi(z)\} = -\int_{z_0}^{z} \frac{\partial}{\partial t} G_1(z-t,\zeta-\zeta_0)\,\psi(t)\,dt \qquad (2.1)$$

$$\Psi^*(z,\zeta) = L^1_{\zeta,z}\{\psi^*(\zeta)\} = -\int_{\zeta_0}^{\zeta} \frac{\partial}{\partial\tau} G_1(z-z_0,\zeta-\tau)\,\psi^*(\tau)\,d\tau$$

into the considerations.

Here $\varphi(z)$, $\psi(z)$, $\varphi^*(\zeta)$ and $\psi^*(\zeta)$ are arbitrary analytic functions of their arguments.

It follows from (2.1) that the following relationships hold:

$$\overset{\circ}{L}_{z,\zeta} = \frac{\partial^2}{\partial z\partial\zeta} L^1_{z,\zeta}, \qquad \overset{\circ}{L}_{\zeta,z} = \frac{\partial^2}{\partial z\partial\zeta} L^1_{\zeta,z} \qquad (2.2)$$

Each of the functions introduced into (2.1) will be a solution of (1.1) for $f(z,\zeta)=0$. By virtue of (1.2), the general solution of the latter can be represented as

$$F(z,\zeta) = \Phi(z,\zeta) + \Phi^*(z,\zeta) + \Psi(z,\zeta) + \Psi^*(z,\zeta) \qquad (2.3)$$

Let us now construct some complete systems of particular solutions of the homogeneous equation (1.1). Let

$$\varphi(z) = \psi(z) = \frac{(z-z_0)^\gamma}{\Gamma(\gamma+1)}, \qquad \varphi^*(\zeta) = \psi^*(\zeta) = \frac{(\zeta-\zeta_0)^\gamma}{\Gamma(\gamma+1)} \qquad (2.4)$$

where we shall as yet assume that $\operatorname{Re}\gamma > -1$.

On the basis of (2.1) we have (2.5)

$$\Phi_\gamma(z-z_0,\zeta-\zeta_0) = \overset{\circ}{L}_{z,\zeta}\left\{\frac{(z-z_0)^\gamma}{\Gamma(\gamma+1)}\right\}, \qquad \Psi_\gamma(z-z_0,\zeta-\zeta_0) = L^1_{z,\zeta}\left\{\frac{(z-z_0)^\gamma}{\Gamma(\gamma+1)}\right\}$$

$$\Phi_\gamma^*(z-z_0,\zeta-\zeta_0) = \Phi_\gamma(\zeta-\zeta_0,z-z_0)$$

$$\Psi_\gamma^*(z-z_0,\zeta-\zeta_0) = \Psi_\gamma(\zeta-\zeta_0,z-z_0)$$

Performing the first two operations in (2.5), and taking account of the formula

$$\int_{z_0}^{z} \frac{(z-t)^k(t-z_0)^\gamma}{\Gamma(k+1)\Gamma(\gamma+1)}\,dt = \frac{(z-z_0)^{k+\gamma+1}}{\Gamma(k+\gamma+2)} \qquad (2.6)$$

we obtain

$$\Phi_\gamma(z-z_0,\zeta-\zeta_0) = \sum_{k=0}^{\infty} \frac{(z-z_0)^{k+\gamma}}{\Gamma(k+\gamma+1)}\, g_k'(\zeta-\zeta_0)$$

$$\Psi_\gamma(z-z_0,\zeta-\zeta_0) = \sum_{k=0}^{\infty} \frac{(z-z_0)^{k+\gamma+1}}{\Gamma(k+\gamma+2)}\, g_k(\zeta-\zeta_0) \qquad (2.7)$$

The remaining solutions are defined in (2.5).

The formulas (2.7) evidently yield an analytic continuation of the solutions onto the whole plane of the parameter γ with the exception of the points $\gamma = -1, -2, \ldots$. In this latter case we obtain by taking into account that the Euler Gamma function is

$$\Gamma(-n) = \infty, \qquad n = 0, 1, 2, \ldots \qquad (2.8)$$

$$\Phi_{-n}(z-z_0,\zeta-\zeta_0) = \sum_{k=0}^{\infty} \frac{(z-z_0)^k}{k!}\, g'_{k+n}(\zeta-\zeta_0) = \frac{\partial^n}{\partial z^n} G_0(z-z_0,\zeta-\zeta_0)$$

$$\Psi_{-n}(z-z_0, \zeta-\zeta_0) = \sum_{k=0}^{\infty} \frac{(z-z_0)^k}{k!} g_{k+n-1}(\zeta-\zeta_0) = \frac{\partial^n}{\partial z^n} G_1(z-z_0, \zeta-\zeta_0)$$

$$\Phi_{-n}^*(z-z_0, \zeta-\zeta_0) = \Phi_{-n}(\zeta-\zeta_0, z-z_0), \Psi_n^*(z-z_0, \zeta-\zeta_0) = \\ = \Psi_{-n}(\zeta-\zeta_0, z-z_0) \qquad (2.9)$$

$$g_{-1}(\zeta) \equiv 0$$

This last result is completely natural since the operators (2. 5) are fractional integrals of order γ of the appropriate kernels, as is easily noted. For $\gamma = n$ ($n = 1, 2 \ldots$) we have n-tuple integrals, for $\gamma = -n$ ($n = 1, 2, \ldots$) nth order derivatives of the kernels.

We call the functions (2. 5) generalized powers since the operators in (2. 5) map the analytic power functions (2. 4) into solutions of (1. 1). This definition is also justified by the fact that generalized powers of Φ_n and Ψ_n have a zero of multiplicity n and $n+2$, respectively, at the point $z = z_0, \zeta = \zeta_0$.

For $\gamma = 0$ we see that the kernels themselves will be generalized constants

$$\Phi_0(z-z_0, \zeta-\zeta_0) = \Phi_0^*(z-z_0, \zeta-\zeta_0) = G_0(z-z_0, \zeta-\zeta_0) \qquad (2.10)$$

$$\Psi_0(z-z_0, \zeta-\zeta_0) = \Psi_0^*(z-z_0, \zeta-\zeta_0) = G_1(z-z_0, \zeta-\zeta_0)$$

The generalized powers defined above will be regular solutions of (1. 1) in any finite simply-connected domain D, D^* ($z \in D$, $\zeta \in D^*$).

3. The functions

$$\Phi_{-n,\psi}(z-z_0, \zeta-\zeta_0) = \sum_{k=0}^{\infty} \frac{(z-z_0)^k}{k!} \psi(k+1) g'_{k+n}(\zeta-\zeta_0)$$

$$\Psi_{-n,\psi}(z-z_0, \zeta-\zeta_0) = \sum_{k=0}^{\infty} \frac{(z-z_0)^k}{k!} \psi(k+1) g_{k+n-1}(\zeta-\zeta_0) \qquad (3.1)$$

$$\Phi^*_{-n,\psi}(z-z_0, \zeta-\zeta_0) = \Phi_{-n,\psi}(\zeta-\zeta_0, z-z_0),$$

$$\Psi^*_{-n,\psi}(z-z_0, \zeta-\zeta_0) = \Psi_{-n,\psi}(\zeta-\zeta_0, z-z_0)$$

$$\psi(k+1) = -C + \sum_{j=1}^{k} \frac{1}{j}$$

are required below.

Here $C = \psi(1)$ is the Euler constant. The functions (3. 1) are analytic in z, ζ in any bounded domain D, D^*.

Now, let us construct particular solutions of (1. 1), which have a logarithmic type singularity or pole of given order at some point.

To do this let us introduce particular solutions by means of the formulas

$$\Theta(z-z_0, \zeta-\zeta_0) = L_{z,\zeta}^{\circ}\{\ln(z-z_0)\}$$

$$\Xi(z-z_0, \zeta-\zeta_0) = L_{z,\zeta}'\{\ln(z-z_0)\}$$

$$\Theta^*(z-z_0, \zeta-\zeta_0) = \Theta(\zeta-\zeta_0, z-z_0)$$

$$\Xi^*(z-z_0, \zeta-\zeta_0) = \Xi(\zeta-\zeta_0, z-z_0) \qquad (3.2)$$

$$\Theta_{-n}(z-z_0,\zeta-\zeta_0)=L_{z,\zeta}{}^{\circ}\{\varphi(z)\},\Xi_{-n}(z-z_0,\zeta-\zeta_0)=L^1{}_{z,\zeta}\{\varphi(z)\}$$
$$\Theta_{-n}{}^*(z-z_0,\zeta-\zeta_0)=\theta_{-n}(\zeta-\zeta_0,z-z_0)$$
$$\Xi_{-n}{}^*(z-z_0,\zeta-\zeta_0)=\Xi_{-n}(\zeta-\zeta_0,z-z_0) \tag{3.3}$$

where

$$\varphi(z)=\frac{(-1)^{n+1}(n-1)!}{(z-z_0)^n}\quad(z\neq z_0),\qquad \varphi(z)=0\quad(z=z_0)$$

Formulas (3.2) define generalized logarithms, and (3.3) yield generalized negative powers. Let us evaluate the generalized logarithm $\Xi(z-z_0,\zeta-\zeta_0)$. On the basis of (1.3) and the binomial formula we have

$$\frac{\partial}{\partial t}G_1(z-t,\zeta-\zeta_0)=-\sum_{k=0}^{\infty}\frac{g_k(\zeta-\zeta_0)}{k!}\sum_{j=0}^{k}(-1)^j C_k^j(z-z_0)^{k-j}(t-z_0)^j$$
$$C_k^j=k!\,/\,j!\,(k-j)! \tag{3.4}$$

Substituting (3.4) into the third of formulas (2.1) and integrating by parts, we obtain, when taking account of (3.3)

$$\Xi(z-z_0,\zeta-\zeta_0)=\Psi_0(z-z_0,\zeta-\zeta_0)\ln[(z-z_0)e^{-C}]-$$
$$-\Psi_{0,\psi}(z-z_0,\zeta-\zeta_0) \tag{3.5}$$

By virtue of (2.2) we have

$$\theta(z-z_0,\zeta-\zeta_0)=\Phi_0(z-z_0,\zeta-\zeta_0)\ln[(z-z_0)e^{-C}]-$$
$$-\Phi_{0,\psi}(z-z_0,\zeta-\zeta_0) \tag{3.6}$$

The remaining generalized logarithms are defined in (3.2).

To determine the generalized negative powers $\Theta_{-n}(z,\zeta)$ we set

$$\chi(z)=\int_{z_0}^{z}\frac{(z-t)^{n-1}}{(n-1)!}\varphi(t)\,dt,\quad \varphi(z)=\chi^{(n)}(z),\quad \chi^{(k)}(z_0)=0\quad(k=0,1,\ldots,n-1) \tag{3.7}$$

in the first of formulas (3.3).

Integrating by parts, we arrive at the expression

$$\theta_{-n}(z-z_0,\zeta-\zeta_0)=$$
$$=\sum_{k=0}^{n}\chi^{(k)}(z)\,g'_{n-k}(\zeta-\zeta_0)+(-1)^{n+1}\int_{z_0}^{z}\frac{\partial^{n+1}}{\partial t^{n+1}}G_0(z-t,\zeta-\zeta_0)\,\chi(t)\,dt \tag{3.8}$$

Taking account of (3.3), (3.1) and (1.3) we obtain from (3.8)

$$\theta_{-n}(z-z_0,\zeta-\zeta_0)=\sum_{k=1}^{n}\frac{(-1)^{k+1}(k-1)!}{(z-z_0)^k}g'_{n-k}(\zeta-\zeta_0)+ \tag{3.9}$$
$$+\Phi_{-n}(z-z_0,\zeta-\zeta_0)\ln[(z-z_0)e^{-C}]-\Phi_{-n,\psi}(z-z_0,\zeta-\zeta_0)\quad(n=1,2,\ldots)$$

The second power is determined analogously

$$\Xi_{-n}(z-z_0,\zeta-\zeta_0)=\sum_{k=2}^{n}\frac{(-1)^k(k-2)!}{(z-z_0)^{k-1}}g_{n-k}(\zeta-\zeta_0)+ \tag{3.10}$$
$$+\Psi_{-n}(z-z_0,\zeta-\zeta_0)\ln[(z-z_0)e^{-C}]-\Psi_{-n,\psi}(z-z_0,\zeta-\zeta_0)\quad(n=2,3,\ldots)$$
$$\Xi_{-1}(z-z_0,\zeta-\zeta_0)=\Psi_{-1}(z-z_0,\zeta-\zeta_0)\ln[(z-z_0)e^{-C}]-$$
$$-\Psi_{-1,\psi}(z-z_0,\zeta-\zeta_0)$$

The remaining generalized negative powers are defined in (3.3). It follows from (3.9) and (3.10) that the order of the principal singularity for the function $\Theta_{-n}(z, \zeta)$ is n, the order of the principal singularity for $\Xi_{-n}(z, \zeta)$ is $n-2$, for $n = 2, 3, \ldots$; there is only the logarithmic singularity for $n = 1, 2$.

From the theory of generalized functions in the Sobolev-Schwartz sense it follows that the function θ_{-n} (Ξ_{-n}) can be defined as the nth power derivative of Θ (Ξ), i.e. that the following relationships hold:

$$\theta_{-n}(z, \zeta) = \frac{\partial^n}{\partial z^n}\theta(z, \zeta), \qquad \Xi_{-n}(z, \zeta) = \frac{\partial^n}{\partial z^n}\Xi(z, \zeta) \tag{3.11}$$

Particular solutions of (1.1) with a singularity at a point z_1, ζ_1 different from z_0, ζ_0 can be obtained analogously. We do not write down these solutions.

The functions Φ_n, Φ^*_n, Ψ_n and $\Psi_n{}^*$ $(n = 0, 1, \ldots)$ form a complete system of particular solutions of the homogeneous equation (1.1) for any bounded simply-connected domain D, D^*. In combination with the functions Θ_{-n}, Θ_{-n}^*, Ξ_{-n}, Ξ_{-n}^* they form a complete system of particular solutions [3] for any doubly-connected domain D, D^*.

4. From the above it follows that the system of particular solutions Φ_γ, $\Phi_\gamma{}^*$, Ψ_γ and $\Psi_\gamma{}^*$ does not possess symmetry with respect to the subscript. For $\gamma = 1, 2, \ldots$ we will have a system of particular solutions as a result of successive integration of the kernels G_0 and G_1; for $\gamma = -1, -2, -, \ldots$ we have solutions which are the result of successive differentiation of the same kernels. Meanwhile, both these functions are regular solutions of (1.1). The circumstance mentioned suggests the idea of constructing that system of solutions which would be symmetric with respect to the subscript in the sense that a sign change in the latter would not go out of the given class of regular solutions.

Let us examine the following convolutions:

$$\Phi_{\lambda,\gamma}(z - z_0, \zeta - \zeta_0) = \int_{\zeta_0}^{\zeta} \frac{(\zeta - \tau)^{\lambda-1}}{\Gamma(\lambda)} \Phi_\gamma(z - z_0, \tau - \zeta_0)\, d\tau \quad (\operatorname{Re}\lambda > 0,\ \operatorname{Re}\gamma > 0)$$

$$\Psi_{\lambda,\gamma}(z - z_0, \zeta - \zeta_0) = \int_{\zeta_0}^{\zeta} \frac{(\zeta - \tau)^{\lambda-1}}{\Gamma(\lambda)} \Psi_{\gamma-1}(z - z_0, \tau - \zeta_0)\, d\tau + \tag{4.1}$$

$$+ \int_{\zeta_0}^{\zeta} \frac{(\zeta - \tau)^{\lambda-2}}{\Gamma(\lambda - 1)} \Psi_\gamma(z - z_0, \tau - \zeta_0)\, d\tau \quad (\operatorname{Re}\lambda > 1,\ \operatorname{Re}\gamma > 1)$$

The functions defined in (4.1) will be solutions of (1.1) for some $f(z, \zeta)$.

Let us elucidate the form of the right side $f(z, \zeta)$ corresponding to these solutions. It is easy to see from (2.1) and (2.5) that

$$\Psi_\gamma(z - z_0, \zeta - \zeta_0) = \int_{z_0}^{z} \frac{(z - t)^{\gamma-1}}{\Gamma(\gamma)} G_1(t - z_0, \zeta - \zeta_0)\, dt \tag{4.2}$$

Substituting (4.2) into the second formula in (4.1) we have

$$\Psi_{\lambda,\gamma}(z - z_0, \zeta - \zeta_0) = \int_{z_0}^{z} dt \int_{\zeta_0}^{\zeta} \left\{ \frac{(\tau - \zeta_0)^{\lambda-1}(t - z_0)^{\gamma-2}}{\Gamma(\lambda)\Gamma(\gamma - 1)} + \right.$$

$$\left. + \frac{(\tau - \zeta_0)^{\lambda-2}(t - z_0)^{\gamma-1}}{\Gamma(\lambda - 1)\Gamma(\gamma)} \right\} G_1(z - t, \zeta - \tau)\, d\tau \tag{4.3}$$

By virtue of (1.7) we hence conclude that the right side of (1.1) corresponding to the solution $\Psi_{\lambda,\gamma}$ is

$$f(z,\zeta)=\frac{(\zeta-\zeta_0)^{\lambda-1}(z-z_0)^{\gamma-2}}{\Gamma(\lambda)\Gamma(\gamma-1)}+\frac{(\tau-\zeta_0)^{\lambda-2}(t-z_0)^{\gamma-1}}{\Gamma(\lambda-1)\Gamma(\gamma)}\tag{4.4}$$

The function $f(z, \zeta)$ corresponding to the solution $\Phi_{\lambda,\gamma}$ has the form

$$f(z,\zeta)=\frac{(\zeta-\zeta_0)^{\lambda-2}(z-z_0)^{\gamma-2}}{\Gamma(\lambda-1)\Gamma(\gamma-1)}\tag{4.5}$$

Integrating in (4.1), and taking account of (2.7) and (1.5) we obtain expressions for the functions $\Phi_{\lambda,\gamma}$ and $\Psi_{\lambda,\gamma}$

$$\Phi_{\lambda,\gamma}(z-z_0,\zeta-\zeta_0)=\sum_{k=0}^{\infty}\frac{(z-z_0)^{k+\gamma}}{\Gamma(k+\gamma+1)}\sum_{s=0}^{\infty}C_{k,s}\frac{(\zeta-\zeta_0)^{s+\lambda}}{\Gamma(s+\lambda+1)}$$

$$\Psi_{\lambda,\gamma}(z-z_0,\zeta-\zeta_0)=\sum_{k=0}^{\infty}\frac{(z-z_0)^{k+\gamma}}{\Gamma(k+\gamma+1)}\sum_{s=0}^{\infty}C_{k,s}\frac{(\zeta-\zeta_0)^{s+\lambda+1}}{\Gamma(s+\lambda+2)}+\tag{4.6}$$

$$+\sum_{k=0}^{\infty}\frac{(z-z_0)^{k+\gamma+1}}{\Gamma(k+\gamma+2)}\sum_{s=0}^{\infty}C_{k,s}\frac{(\zeta-\zeta_0)^{s+\lambda}}{\Gamma(s+\lambda+1)}$$

$$C_{2k,2s}=a_{k,s},\quad C_{2k+1,2s}=C_{2k,2s+1}=0,\quad C_{2k+1,2s+1}=b_{k,s}$$

Formulas (4.6) yield an analytic continuation of the integrals in (4.1) to complex values of the subscripts λ and γ, with the exception of the points $\gamma=-1,-2,\ldots$; $\lambda=-1,-2,-,\ldots$. If one of the subscripts is arbitrary, and the other takes on negative integer values, then the functions (4.6) yield solutions of (1.1) with right side $f(z,\zeta)=0$.

Let $\gamma=m, \lambda=-n\ (m, n=1,2,\ldots)$, keeping in mind (2.8) we then obtain

$$\Phi_{-n,m}(z-z_0,\zeta-\zeta_0)=\frac{\partial^n}{\partial\zeta^n}\int_{z_0}^{z}\frac{(z-t)^{m-1}}{(m-1)!}G_0(t-z_0,\zeta-\zeta_0)\,dt\tag{4.7}$$

Analogous relationships hold also for $\Psi_{-n,m}(z,\zeta)$. Symmetry relationships follow from (4.1) or (4.6)

$$\Phi_{\lambda,\gamma}(z,\zeta)=\Phi_{\gamma,\lambda}(z,\zeta)=\Phi_{\gamma,\lambda}(\zeta,z),\quad \Psi_{\lambda,\gamma}(z,\zeta)=\Psi^*_{\gamma,\lambda}(z,\zeta)=\Psi_{\gamma,\lambda}(\zeta,z)\tag{4.8}$$

If $\lambda=-n,\ \gamma=n\ (n=0,1,2,\ldots)$, then according to (4.8) we have

$$\Phi^*_{-nn}(z,\zeta)=\Phi_{n-n,}(\zeta,z),\quad \Psi^*_{n,n}(z,\zeta)=\Psi_{n,n}(\zeta,z)\tag{4.9}$$

Therefore, the constructed system of solution with integer subscript n $\Phi_{-n,n}(z,\zeta)$, $\Phi^*_{-n,n^*}(z,\zeta)$, $\Psi_{-n,n}(z,\zeta)$ and $\Psi_{-n,n^*}(z,\zeta)$ is symmetric in the subscript in the above mentioned sense. We call these solutions – solutions of the first kind with integer subscript. They can be represented as

$$\Phi_{-n,n}(z-z_0,\zeta-\zeta_0)=\frac{\partial^n}{\partial\zeta^n}\int_{z_0}^{z}\cdots\int_{z_0}^{z}G_0(t-z_0,\zeta-\zeta_0)\,dt^n\quad(n=0,1,\ldots)$$

$$\Psi_{-n,n}(z-z_0,\zeta-\zeta_0)=\frac{\partial^n}{\partial\zeta^n}\int_{z_0}^{z}\cdots\int_{z_0}^{z}G_1(t-z_0,\zeta-\zeta_0)\,dt^{n-1}+\tag{4.10}$$

$$+\frac{\partial^{n+1}}{\partial\zeta^{n+1}}\int_{z_0}^{z}\cdots\int_{z_0}^{z}G_1(t-z_0,\zeta-\zeta_0)\,dt^n\qquad(n=1,2,\ldots)$$

$$\Psi_{0,0}(z-z_0,\ \zeta-\zeta_0)=\left(\frac{\partial}{\partial\zeta}+\frac{\partial}{\partial z}\right)G_1(z-z_0,\ \zeta-\zeta_0)=\frac{\partial}{\partial\zeta}G_1(z-z_0,\ \zeta-\zeta_0)$$

5. Let us illustrate how solutions of the first kind with integer subscript "work" by means of the example of a cylindrical shell. In this case $\delta = 1$ and formulas (1.8) become

$$G_k(z,\zeta)=\frac{1}{2\pi i}\int_L e^{p\zeta}G^*_k(p,z)\,dp \tag{5.1}$$

$$G_1^*(p,z)=\frac{1}{2p^2}\left(\exp\frac{pz}{p-1}-\exp\frac{-pz}{p+1}\right),\qquad G_0^*(p,z)=p\frac{\partial}{\partial z}G_1^*(p,z)$$

(L is a closed contour enclosing the points $p=\pm 1$)

By virtue of (5.1) we have from the first formula in (4.10)

$$\Phi_{-n,n}(z,\zeta)=\frac{1}{2\pi i}\int_L e^{p\zeta}\left\{\frac{(p-1)^{n-1}}{2}\exp\frac{pz}{p-1}+(-1)^n\frac{(p+1)^{n-1}}{2}\exp\frac{-pz}{p+1}\right\}dp \tag{5.2}$$

Let us transform this latter formula into

$$\Phi_{-n,n}(z,\zeta)=\frac{e^{z+\zeta}}{2\pi i}\int_L\frac{(p-1)^{n-1}}{2}\exp\left((p-1)\zeta+\frac{z}{p-1}\right)dp+$$
$$+\frac{e^{-z-\xi}}{2\pi i}\int_L(-1)^n\frac{(p+1)^{n-1}}{2}\exp\left((p+1)\zeta+\frac{z}{p+1}\right)dp \tag{5.3}$$

We hence easily obtain

$$\Phi_{-n,n}(z,\zeta)=\left(\frac{z}{\zeta}\right)^{n/2}I_n(2\sqrt{z\zeta})\,\mathrm{ch}\left(z+\zeta-\frac{i\pi n}{2}\right)\qquad(n=0,1,\ldots) \tag{5.4}$$

We obtain the remaining solution with integer subscript analogously

$$\Phi^*_{-n,n}(z,\zeta)=\Phi_{-n,n}(\zeta,z)=(\zeta/z)^{n/2}I_n(2\sqrt{z\zeta})\,\mathrm{ch}\,(z+\zeta-{}^1/_2\,i\pi n)$$
$$\Psi_{-n,n}(z,\zeta)=(z/\zeta)^{n/2}I_n(2\sqrt{z\zeta})\,\mathrm{sh}\,(z+\zeta-{}^1/_2\,i\pi n)$$
$$\Psi^*_{-n,n}(z,\zeta)=\Psi_{-n,n}(\zeta,z)=(\zeta/z)^{n/2}I_n(2\sqrt{z\zeta})\,\mathrm{sh}\,(z+\zeta-{}^1/_2\,i\pi n) \tag{5.5}$$

where $I_n(t)$ is the modified Bessel function of the first kind.

We have the following expressions for the kernel $G_0(z,\zeta)$ and the Riemann function $G_1(z,\zeta)$

$$G_0(z,\zeta)=I_0(2\sqrt{z\zeta})\,\mathrm{ch}\,(z+\zeta) \tag{5.6}$$

$$G_1(z,\zeta)=\int_0^{\xi}I_0(2\sqrt{z\zeta})\,\mathrm{sh}\,(z+\zeta)\,d\xi,\qquad 2\xi=z+\zeta$$

The functions (5.4) and (5.5) agree with the known regular solutions in the theory of a circular cylindrical shell.

6. Let us write the fundamental solution of (1.1). To do this, we put [3] in the representations (1.2) the following:

$$\varphi_0(z)=\psi_0(\zeta)=0,\quad \varphi_1(z)=A\ \ln(z-z_0),\quad \psi_1(\zeta)=A\ \ln(\zeta-\zeta_0) \tag{6.1}$$

It is easy to see from (3.2) and (2.1) that the fundamental solution is

$$\Omega(z-z_0,\ \zeta-\zeta_0)=A\{\Xi(z-z_0,\ \zeta-\zeta_0)+\Xi^*(z-z_0,\ \zeta-\zeta_0)\} \tag{6.2}$$

or by virtue of (3.5) and (2.10)

$$\Omega(z-z_0,\ \zeta-\zeta_0)=2AG_1(z-z_0,\ \zeta-\zeta_0)\ln\frac{\sqrt{(z-z_0)(\zeta-\zeta_0)}}{\exp C}$$
$$-A\,[\Psi_{0,\psi}(z-z_0,\ \zeta-\zeta_0)+\Psi_{0,\psi}(\zeta-\zeta_0,\ z-z_0)] \tag{6.3}$$

By virtue of (1.3), (1.5) and (1.1) the function $\Omega(z, \zeta)$ has a singularity of type $\rho^2 \ln \rho$ at the point $z = z_0$, $\zeta = \zeta_0$, where $\rho^2 = \sqrt{(x - x_0)^2 + (y - y_0)^2}$. Evidently any solution regular at the point $z = z_0$, $\zeta = \zeta_0$ can be added to the function (6.3).

7. The application of potential theory in shell theory is based on formulas of Darboux type (see [3]) which yield a representation of the solution within a domain in terms of the fundamental solution and values of the solution and its first three derivatives on the domain contour. Later investigations [5] showed the effectiveness of applying potential theory in many boundary value problems of shallow shell theory.

Let us represent the Darboux formula for our case in the following two ways:

$$\text{a)} \quad F(x, y) = A^* \int_L \{N^*(\Omega, F) - N^*(F, \Omega)\} ds \left(A^* = \frac{ia^2}{16\pi A\beta^2}\right) \tag{7.1}$$

$$\Omega = \Omega(x' - x, y' - y), \quad F = F(x', y'), \quad x', y' \in L, \quad x, y \in D$$

$$N^*(u, v) = u \frac{\partial}{\partial n} \nabla^2 v - \frac{\partial u}{\partial n}\left[\frac{\partial^2 v}{\partial n^2} - \frac{\partial \theta}{\partial n}\frac{\partial v}{\partial s} + (b\cos^2\theta + b^*\sin^2\theta) v\right] -$$
$$- \frac{\partial u}{\partial s}\left[\frac{\partial^2 v}{\partial n\, \partial s} - \frac{\partial \theta}{\partial s}\frac{\partial v}{\partial s} + v\frac{b^* - b}{2}\sin 2\theta\right], \quad b^* = \alpha b, \quad b = \frac{16 i\beta^2}{(1-\alpha)a_2}$$

Here L is the domain boundary, n and s the directions normal and tangent to the domain contour, respectively, θ the angle between the x-axis and the external normal, D the domain included within L, and A is taken so that $A^* = 1$.

$$\text{b)} \quad F(z, \zeta) = \frac{i}{2}\int_L \left\{N(\Omega, F)\frac{d\tau}{ds} - N^*(\Omega, F)\frac{dt}{ds}\right\} ds \tag{7.2}$$

$$N(F_1, F) = \frac{1}{2} F_1 \frac{\partial^3 F}{\partial t\, \partial \tau^2} - \frac{1}{2} F_2 \frac{\partial^3 F_1}{\partial t\, \partial \tau^2} - F_1\left(\delta\frac{\partial F}{\partial \tau} + \frac{\partial F}{\partial t}\right) +$$
$$+ F\left(\delta\frac{\partial F_1}{\partial \tau} + \frac{\partial F_1}{\partial t}\right) - \frac{1}{4}\left(\frac{\partial F_1}{\partial t}\frac{\partial^2 F}{\partial \tau^2} + \frac{\partial F_1}{\partial \tau}\frac{\partial^2 F}{\partial t\, \partial \tau}\right) + \frac{1}{4}\left(\frac{\partial F}{\partial t}\frac{\partial^2 F_1}{\partial \tau^2} + \frac{\partial F}{\partial \tau}\frac{\partial^2 F_1}{\partial t\, \partial \tau}\right)$$

$$N^*(F_1, F) = \frac{1}{2} F_1 \frac{\partial^3 F}{\partial t^2\, \partial \tau} - \frac{1}{2} F \frac{\partial^3 F_1}{\partial t^2\, \partial \tau} - F_1\left(\delta\frac{\partial F}{\partial t} + \frac{\partial F}{\partial \tau}\right) + F\left(\delta\frac{\partial F_1}{\partial t} + \frac{\partial F_1}{\partial \tau}\right) -$$
$$- \frac{1}{4}\left(\frac{\partial F_1}{\partial t}\frac{\partial^2 F}{\partial t\, \partial \tau} + \frac{\partial F_1}{\partial \tau}\frac{\partial^2 F}{\partial t^2}\right) + \frac{1}{4}\left(\frac{\partial F}{\partial t}\frac{\partial^2 F_1}{\partial t\, \partial \tau} + \frac{\partial F}{\partial \tau}\frac{\partial^2 F_1}{\partial t^2}\right)$$

$$\Omega = \Omega(t - z, \tau - \zeta), \quad F = F(t, \tau), \quad t, \tau \in L, L^*, \quad z, \zeta \in D, D^*$$

It is convenient to use (7.1) to reduce boundary value problems of shell theory to integral equations. After some manipulations, (7.2) reduces to a representation of the solution of (1.1) in terms of generalized Cauchy type integrals and some others, whose kernels are solutions of the type

$$\theta_{-n}(z, \zeta), \quad \theta_{-n}^*(z, \zeta), \ \Xi_{-n}(z, \zeta), \Xi_{-n}^*(z, \zeta), \ (n = 0, 1)$$

8. In solving boundary value problems by series, it is conveneient to transform the particular solutions written in the form (2.7), (2.9), (3.9), (3.10), etc., into Fourier type series in θ. This can be done by using the easily deducible formula

$$\sum_{k=0}^{\infty} \frac{z^{2k+\gamma} A_{2k+m}}{\Gamma(2k+\gamma+1)} \sum_{s=0}^{\infty} \frac{\zeta^{2s+\nu} B_{k+r, s+q}}{\Gamma(2s+\nu+1)} = z^{\gamma}\zeta^{\nu}\left\{\sum_{j=0}^{\infty} \frac{A_{m+2j} B_{r+j, q+j} (z\zeta)^{2j}}{\Gamma(2j+\gamma+1)\Gamma(2j+\nu+1)} +\right.$$
$$+ \sum_{k=1}^{\infty}\left(\frac{z}{\zeta}\right)^k \sum_{j=0}^{\infty} \frac{A_{m+2j+2k} B_{j+k+r, j+q} (z\zeta)^{2j+k}}{\Gamma(2j+2k+\gamma+1)\Gamma(2j+\nu+1)} + \tag{8.1}$$

$$+\sum_{k=1}^{\infty}\left(\frac{\zeta}{z}\right)^k\sum_{j=0}^{\infty}\frac{A_{m+2j}B_{r+j,\,j+q+k}(z\zeta)^{2j+k}}{\Gamma(2j+\gamma+1)\Gamma(2j+2k+\nu+1)}\Bigg\}$$

In applying the mentioned procedure to the obtained particular solutions the relationships to which the quantities $a_{k,s}$ and $b_{k,s}$ in (1.5) are subject must be kept in mind

$$a_{r,\,r+k}(\delta)=\alpha_1^{-2r}\varkappa_{2r,\,2r+k},\qquad b_{r,\,r+k}(\delta)=\alpha_1^{-2r-1}\varkappa_{2r+1,\,2r+k+1} \tag{8.2}$$

where

$$\varkappa_{r,\,r+k}=\sum_{j=0}^{r}2^{2r-2j}\left(-\frac{2}{\delta+1}\right)^j\frac{(r+k)!\,(2j+2k)!}{j!\,(j+k)!\,(j+2k)!\,(r-j)!},\qquad \alpha_1=1-\alpha=\frac{2}{1+\delta}$$

In conclusion, let us note that the parameter $0\leqslant\delta\leqslant\infty$. If the radii of shell curvature are $R>0$, $R_1>0$, $(R_1\geqslant R)$, then $1\leqslant\delta\leqslant\infty$; if $R>0$, $R_1<0$ and $|R_1|\geqslant|R|$, then $0\leqslant\delta\leqslant 1$. The values $\delta=1$ and ∞ correspond to cylindrical and spherical shells. The value $\delta=0$ corresponds to a shell of hyperbolic type for $|R|=|R_1|$ (pseudosphere).

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OPTIMAL STRATEGIES IN A LINEAR DIFFERENTIAL GAME

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The game problem of bringing onto a prescribed set a controlled object whose motion is described by linear differential equations is considered. The conditions under which a saddle point exists in the class of generalized strategies in the differential game under investigation are derived. A procedure for constructing the players' generalized optimal strategies is proposed.

1. Let us consider the game problem of bringing onto the prescribed set M a controlled object whose motion is described by the system of differential equations

$$\frac{dx}{dt}=A(t)x+B(t)u-C(t)v \tag{1.1}$$